Erdős 1947: If $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ there exists a two coloring of the edges of $K_n$ with no monochromatic $K_k$.

Proof: Color Randomly!
Theorem (Turán). Any graph $G$ has an independent set $S$ with

$$|S| \geq \sum_{v \in G} \frac{1}{d_v + 1}$$

Randomized Algorithm

- Order Vertices Randomly.
- Place $v$ in $S$ greedily.

If $v$ comes before its neighbors then it goes into $S$.

$$\Pr[v \in S] \geq \frac{1}{d_v + 1}$$

Linearity of Expectation:

$$E[|S|] = \sum_{v \in G} \Pr[v \in S] \geq \sum_{v \in G} \frac{1}{d_v + 1}$$

Erdős Magic: Such $S$ **MUST** exist.
$|A_i| = n, \ 1 \leq i \leq m = 2^{n-1}k$

Seek Red/Blue $\chi$ with no $A_i$ monochromatic

Erdős [1963]: $k < 1 \Rightarrow \exists \chi$

Beck [1978]: $k < cn^{1/3} \Rightarrow \exists \chi$

Radhakrishnan-Srinivasan[2000]

$$k < c[n/\ln n]^{1/2} \Rightarrow \exists \chi$$

Erdős [1964]: There exists family with $k = cn^2$

with no $\chi$
Coloring Algorithm(s)

1. Color Randomly
2. Order Vertices Randomly.
3. Consider sequentially. If $v$ “still dangerous” switch $\chi(v)$ with probability $p$

Still Dangerous: $v \in A_i$ which has always been monochromatic

FAIL: Some $A_i$ monochromatic at end

Erdős Magic: If $\Pr[FAIL] < 1$ $\chi$ MUST exist
Two Failure Modes

FAILI: $A_i$ was “Red” and stayed Red

FAILII: $A_i$ wasn’t Red and became Red

\[ \Pr[FAILI(A_i)] = 2^{1-n}(1 - p)^n \]

\[ \Pr[FAILII] \leq (2^{n-1}k)(2^{1-n}(1 - p)^n) = k(1 - p)^n \]
$A_i$ blames $A_j$ if

- $A_i \cap A_j = \{v\}$
- $A_j$ Blue at start
- $A_i$ Red at end
- $v$ LAST point of $A_i$ to change
- When $v$ reached $A_j$ all Blue

Theorem:

If FAILII then some $A_i$ blames some $A_j$

Corollary:

$$\Pr[FAILII] \leq \sum_{i \neq j} \Pr[A_i \text{ blames } A_j]$$
Bounding $\Pr[A_i \text{ blames } A_j]$ 

Fix ordering.

- Factor 2 for Red/Blue symmetry
- $v$ Blue and Flips: $p/2$
- $w \in A_j$ after $v$: $1/2$
- $u \in A_i$ after $v$: $1/2$
- $w \in A_j$ before $v$: $1/2 - p/2$
- $u \in A_i$ before $v$: $1/2 + p/2$

$I$: Number of $w \in A_i$ before $v$

$J$: Number of $u \in A_j$ before $v$

$\Pr[A_i \text{ blames } A_j | I, J] = 2^{2-2n}p(1+p)^{|I|}(1-p)^{|J|}$
A Bad Gamble

\( n - 1 \) Red Cards, \( n - 1 \) Blue Cards, Joker
Shuffle. Start with 1000 Euro
Red: Multiply funds by \( 1 + p \)
Blue: Multiply funds by \( 1 - p \)
Joker: Cash In.
Theorem: Expectation less than initial
Corollary

\[ \Pr[A_i \text{ blames } A_j] \leq 2^{2-2np} \]

Corollary

\[ \Pr[FAILII] \leq (2^{n-1}k)^2 2^{2-2np} = k^2p \]
Asymptotic Calculus

\[ \Pr[FAIL] < k(1 - p)^n + k^2p \]

Erdős Magic: If for some \( p \in [0, 1] \)

\[ k(1 - p)^n + k^2p < 1 \quad (*) \]

then \( \chi \) **MUST** exist.

What is \( \max k = k(n) \) so that (*) holds for some \( p \in [0, 1] \)?

**Answer:** \( k \sim c\sqrt{n/\ln n} \)
Liar Game

Paul seeks $x \in \{1, \ldots, 100\}$.

Ten Queries. Carole may lie once.

Theorem: Carole Wins!

Carole plays randomly

At end of game:

$\Pr[x \text{ possible}] = \frac{11}{1024}$

Expected number of possible $100 \cdot \frac{11}{1024} > 1$

When $> 1$ possible Carole wins

Carole sometimes wins

Erdős Magic: Carole always wins!
Counting Connected Graphs
and the Giant/Dominant Component

Joint with
Remco van der Hofstad (EURRANDOM)
Nitin Arora (Courant)
Complexity = E-V+1

\[ C(k, l) = \text{Number of} \]

CONNECTED Labelled Graphs

\[ k \text{ Vertices} \]

Complexity \( l \)

\[ C(k, 0) = k^{k-2} \text{ Cayley} \]

\[ C(k, l) \sim c_l k^{3l/2} k^{k-2} \text{ Wright} \]

\[ l > \left( \frac{1}{2} + \epsilon \right) k \ln k \text{ Erdős-Rényi} \]

\[ k, l \to \infty \text{ Bender, Canfield, McKay} \]
The Łuczak Gem

\[ X = \text{number of } (k, l) \text{ components in } G(n, p) \]

\[ E[X] = C(k, l) \binom{n}{k} p^{k+l-1} (1-p)^{k(n-k)} + \binom{k}{2} -(k+l-1) \]

\[ X \leq \frac{n}{k} \text{ tautologically} \]

\[ C(k, l) \leq \frac{n}{k} \left[ \binom{n}{k} p^{k+l-1} (1-p)^{k(n-k)} + \binom{k}{2} -(k+l-1) \right] - 1 \]

for all \( n, p \).

Minimize using Asymptotic Calculus!

If \( l = \Theta(k) \), \( p \sim cn^{-1} \), \( c > 1 \).

Pick \( n, p \) so Giant most likely \( (k, l) \)

\[ E[X] = \Omega(n^{-2}) \] (even better)

\( C(k, l) \) within \( n^2 \) Factor
Tilted Balls in Bins

$k - 1$ balls, $k$ bins, $p \in (0, 1]$

Truncated Geometric

Ball $j$ in Bin $T_j$

$$Pr[T_j = i] = \frac{p(1 - p)^{i-1}}{1 - (1 - p)^k}$$

$Z_i$ balls in bin $i$

$Y_0 = 1$, $Y_i = Y_{i-1} + Z_i - 1$ (so $Y_k = 0$)

TREE: $Y_t > 0$, $0 \leq t < k$

$$M := \sum_{i=0}^{k} (Y_i - 1) = \binom{k}{2} - \sum_{j=1}^{k} T_j$$
$G(k, p)$ Random Graph
Vertices $0, 1, \ldots, k - 1$
Adjacency Prob $p$

THM: Prob $G(k, p)$ Connected & Complexity $l$

$$l = C(k, l)p^{k+l-1}(1-p)^{k(k-1)/2-k-l+1} = A_1 A_2 A_3$$

with

$$A_1 = (1 - (1 - p)^k)^{k-1}$$

$$A_2 = \Pr[\text{TREE}]$$

$$A_3 = \Pr[\text{BIN}[M, p] = l | \text{TREE}]$$

Strategy: $A_2, A_3$ determine $C(k, l)$
Breadth First Search

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
N & N & Y & Y & N \\
N & N & - & - & N \\
Y & N & - & - & N \\
- & Y & - & - & Y \\
- & - & - & - & - \\
- & - & - & - & - \\
\end{array}
\]

\[T_3 = T_4 = 1, \ T_1 = 3, \ T_2 = T_5 = 4\]

\[A_1: \ \text{All } T_j \ \text{defined}\]

\[\vec{Z} = (2, 0, 1, 2, 0, 0)\]

\[\vec{Y} = (1, 2, 1, 1, 2, 1, 0)\]

TREE: BFS doesn’t terminate early

Tree Edges 03, 04, 41, 12, 15

\[M = 2 \ \text{Unexposed 34, 25}\]
Setting the Tilt $p$

\[ \mu := E[M], \quad \sigma^2 := Var[M] \]

\[ p\mu = l \]

Three Regimes

Small $l = o(k), \; k^{-3/2} \ll p \ll k^{-1}$

Large $l = \Theta(k), \; p = \Theta(k^{-1})$

Very Large $l \gg k, \; p \gg k^{-1}$

($l > ck \ln k; \; p > c' \frac{\ln k}{k}$ Erdős-Rényi)
Small: $k^{-3/2} \ll p \ll k^{-1}$

$\epsilon = \frac{1}{2} pk$

Left $Z_i \text{ Poisson } 1 + \epsilon$

Galton-Watson $\Pr[\text{ESC}] \sim 2\epsilon$

Right $Z_i^* = Z_{k-i}; Y_i^* = Y_{k-i}$

$Y_0^* = 0, Y_i^* = Y_{i-1}^* + 1 - Z_i^*$

$Z_i^*$ Poisson $1 - \epsilon$

$\Pr[\text{ESC}^*] \sim \epsilon$

*** Scaling for ESC, ESC$^*$ is $\epsilon^{-2} \ll k$

$\Pr[\text{TREE}] \sim \Pr[\text{ESC}] \Pr[ESC^*] \sim 2\epsilon^2$
Large: \( p \sim \frac{c}{k} \)

Left \( Z_i \) Poisson \( \frac{c}{1-e^{-c}} \)

Galton-Watson \( \Pr[\text{ESC}] \sim 1 - e^{-c} \)

Right \( Z_i^* = Z_{k-i} \), \( Y_i^* = Y_{k-i} \)

\( Y_0^* = 0 \), \( Y_i^* = Y_{i-1}^* + 1 - Z_i^* \)

\( Z_i^* \) Poisson \( \frac{ce^{-c}}{1-e^{-c}} \)

\( \Pr[\text{ESC}^*] \sim 1 - \frac{ce^{-c}}{1-e^{-c}} \)

Chernoff: \( Y_i > 0 \) in middle

\( \Pr[\text{TREE}] \sim \Pr[\text{ESC}] \Pr[\text{ESC}^*] \rightarrow 1 - (c+1)e^{-c} \)

Very Large \( p \gg k^{-1} \)

Chernoff: \( Y_i > 0 \) all \( i \)

\( \Pr[\text{TREE}] \rightarrow 1 \)
Gaussian $M$

$$M := \binom{k}{2} - \sum_{j=1}^{k} T_j$$

Esseen: $\Pr[M < \mu + u\sigma] \rightarrow \Pr[N(0, 1) < u]$

*** Still holds conditional on TREE

Hardest when $p$ barely $\gg k^{-3/2}$

Easy when $p$ Large

Trivial when $p$ Very Large
From CLT to Local Stats

\[ E[W_k] = \mu_k, \quad \text{Var}[W_k] = \sigma_k^2, \quad V_k = BIN[W_k, p_k], \]

\[ l_k = \mu_k p_k = E[V_k] \]

Assume \( p_k^2 \sigma_k^2 = O(p_k \mu_k) \)

\( (\sigma_k^+)^2 := p_k^2 \sigma_k^2 + p_k \mu_k \)

Assume \( \sigma_k^{-1} (W_k - \mu_k) \to N(0, 1) \)

**THM:** \( V_k \) Local CLT, Mean \( l_k \), Var \( (\sigma_k^+)^2 \)

\[ \Pr[V_k = l_k] \to \frac{1}{\sqrt{2\pi \sigma_k^+}} \]

Apply to \( M|\text{TREE} \)
\[ A_3 = \Pr[BIN[M, p] = l | TREE] \]

Small: \( p \mu = l = p^2 \sigma^2, \ A_3 \sim (4\pi l)^{-1/2} \)

Very Large: \( p \mu = l \gg p^2 \sigma^2, \ A_3 \sim (2\pi l)^{-1/2} \)

Large: \( p \sim \frac{c}{k}. \ p^2 \sigma^2 = \Theta(l), \ A_3 \sim g(c) l^{-1/2} \)
The Giant/Dominant Component

\[ G(n, p) \]

\[ p = \frac{c}{n}, \ c > 1 \text{ Erdős-Rényi Giant} \]

\[ p = \frac{1}{n} + \lambda n^{-4/3}, \ \lambda \to +\infty \]

Supercritical: Dominant Component

THM: Probability \( C(v) \) has \( k \) vertices, complexity \( l \) is \( \sim A^*_1 A_2 A_3 \) with

\[ A^*_1 = \Pr[\text{BIN}\[n - 1, 1 - (1 - p)^k\] = k - 1] \]

Corollary: Local Stats for \( k, l \) of Giant/Dominant Component. Correlated Gaussians

(Caution: Work in Progress!)
Generating Random Connected Graph

Time $\Theta(K + L)$ (!!) for $L = \Omega(\ln K)$

$p$ with $p \mu = L$

Tilted Balls into Boxes

$L = \Omega(K)$ get BFS Tree with prob. $\Omega(1)$

$L = o(K)$ use Fast Abort.

Add precisely $L$ of $M$ unexposed with prob.

$$\frac{\Pr[\text{BIN}[M, p] = L]}{\max_m \Pr[\text{BIN}[m, p] = L]}$$
Games Mathematicians Play
Paul versus Carole

$N$ Possibilities

$Q$ Yes/No Paul Queries

$K$ (or fewer) Carole Lies

Try it with $N = 100$, $Q = 10$, $K = 1$

Carole plays Adversary Strategy

$\Rightarrow$ Perfect Information

$\Rightarrow$ Winning Strategy for Paul or Carole

$B_K(Q) = \text{maximal } N \text{ so that Paul Wins}$

Theorem:

$$B_K(Q) \sim \frac{2^Q}{\binom{Q}{K}}$$
Carole Strategy

Notation

\[ \binom{Q}{\leq K} = \sum_{I=0}^{K} \binom{Q}{I} \]

Theorem: \( N\left(\binom{Q}{\leq K}\right) > 2^Q \Rightarrow \text{Carole Wins} \)

Proof 1: Preserve Ministrategies

Proof 2: Random Play

Proof 1 \Rightarrow \text{Proof 2: Derandomization}
Paul Strategy

\( K = 1 \), General Case similar

Weight = Number Viable Ministrategies

Initial Weight \( W_Q > (1 + \epsilon)2^Q \)

Paul splits ministrategies as evenly as possible

\[ W_{i-1} \leq \frac{1}{2}(W_i + i + 1) \]

(worst case: \((2L + 1, 0)\))

Errors don’t accumulate!

When reach \((1, S')\), Endgame
Halflie: No False Negatives

$N$ Possibilities

$Q$ Queries

$K$ Halflies

$A_K(Q) = \text{maximal } N$, Paul Wins

Theorem (Cicalese/Mundici): $A_1(Q) \sim 2^{Q+1}/Q$

Dumitriu/JS:

$$A_K(Q) \sim 2^K B_K(Q) \sim 2^K \frac{2^Q}{\binom{Q}{K}}$$
Position $\vec{x} = (x, y) ((x_0, \ldots, x_K))$

Paul Query: $(a, b) ((a_0, \ldots, a_K))$

Yes $(a, b + x - a)$; No $(x - a, y - b)$

Perfect Split $\left( \frac{x}{2}, \frac{y}{2} - \frac{x}{4} \right)$

Yes/No $L\vec{x} := \left( \frac{x}{2}, \frac{y}{2} + \frac{x}{4} \right)$

Problems: Integrality, Nonnegativity

Weight $W_Q(\vec{x}) = L_Q(\vec{x}) \cdot 1$

$W_Q(x, y) = 2^{-Q}(x(1 + \frac{Q}{2}) + y)$

$2^{-Q}(x_0p_K(Q) + \ldots + x_{K-1}(1 + \frac{Q}{2}) + x_K)$
Paul Strategy

Start $(N, 0), N < (1 - \epsilon)2^{Q+1}/Q$

- Give Ground to $(N, N)$

$$T := \lceil \lg N \rceil$$

- Roundoff so $2^T|N$

- $T$ perfect splits to $L^T(N\vec{1})$

- Endgame: Win in $R$ from
  $(0, 2^R); (1, 2^R - 1); (2, 2^R - 3); (3, 2^R - 5)$
A Combinatorial Approach

1-Set: Subset of \( \{Y, N\}^Q \) with

- **stem**: \( YNNYNYY \)
- **child**: \( YYYNNYN \)
- **child**: \( YNYYYYN \)
- **child**: \( YNNYYYN \)

0-Set: Any Singleton

\( K \)-Set: Depth \( K \) tree with marked “lies.”

- **parent**: \( YYYNNYN \)
- **child**: \( YYYNYNN \)
- **grandchild**: \( YYYNYYYY \)

Theorem: Paul Wins from \((x_0, \ldots, x_K)\) in \( Q \)
\[ \Leftrightarrow \text{Can Pack } x_i \text{ } K - i \text{-Sets in } \{Y, N\}^Q \]
Bound Packing of $K$-Sets

- When all have $\geq LN$, Size $> \binom{L}{\leq K}$

$L \sim \frac{Q}{2}$ Volume Bound $2^Q/\binom{Q/2}{K}$

$o(2^Q Q^{-K})$ have any $L < (1 - o(1)) \frac{Q}{2}$

$A_K(Q) < (1 + o(1)) 2^Q/\binom{Q/2}{K}$

Careful Cutoff

Set $L = \frac{Q}{2} + c \sqrt{Q} \sqrt{\ln Q}$

$A_K(Q) \leq \frac{2^Q}{\binom{Q/2}{K}} (1 + cQ^{-1/2} \sqrt{\ln Q})$

Yan/JS: Remove $\sqrt{\ln Q}$
Two Batch Strategy

Deppe/Ahlswede/Cicalese/Mundici/Dumitriu/JS

\{Y, N\}^{r*}: Number Y within \(r^{0.6}\) of \(\frac{r}{2}\)

\(|\{Y, N\}^{r*}| \sim 2^r\)

"Assume" \(N = |\{Y, N\}^{r*}| \sim 2^Q/(2Q)\)

Associate \(\sigma \in \{Y, N\}^{r*}\) with possibility

Batch 1: \(1 \leq i \leq r\): Is \(\sigma_i = Y\) ?

Carole must say No about half the time!

Endgame from \((1, \sim \frac{r}{2})\) in One Batch
Arbitrary Channel

$T$-ary queries

$E$ lie patterns

Example with $T = 3$, $E = 4$

Ternary Answers A/B/C.

Carole may lie B for A, A for B, A or B for C.

Theorem (Dumitriu, JS):

\[ A^*_K(Q) \sim \frac{T^K T^Q}{E^K \binom{Q}{K}} \]
Open Question

What is the maximum number $G(R)$ of disjoint 1-Shadows in $\{Y, N\}^R$?

$$\frac{2^R}{R+1} \leq G(R)$$

$$G(R) \leq 2^R \frac{2^R}{R}(1 + o(1))$$

Asymptotic Factor of Two Gap.
Jim Propp’s
Random Walk
Simulator
The $P$-Machine on $\mathbb{Z}$

Initially: Arbitrary chips on even positions

Every position $x$ has “arrow” $\epsilon_x = \pm 1$.

Initially: All $\epsilon_x$ arbitrary

Each round every chip moves one position

If $2a$ at $x$ then $a$ to $x \pm 1$.

If $2a + 1$ at $x$ then $a$ to $x \pm 1$.

Then “odd” chip to $x + \epsilon_x$ and,

critically, reset $\epsilon_x \leftarrow -\epsilon_x$. 
The $L$-machine

Chips infinitely divisible

If $b$ at $x$ then $\frac{1}{2}b$ to $x \pm 1$.

Chips at $(x, t)$ is Expected Number if every chip takes Random Walk

Fix initial start, arrows

$P(x, t)$ chips at $(x, t)$ in $P$-machine

$L(x, t)$ chips at $(x, t)$ in $L$-machine

Theorem:

$$|P(x, t) - L(x, t)| \leq 3$$
Generalizations

Any bipartite graph $G$ with Finite Degrees

Initially: All chips on “even” positions

Each $x$ has arrow toward neighbor

For each $x$ ordering of neighbors.

Put “extra” $i$ chips to next $i$ neighbors

And, critically, readjust arrow

\textit{Sometimes}

\[ |P(x,t) - L(x,t)| \leq K_G \]

\textit{JS/Cooper: Yes for $Z^d$}
Outline of Argument (for $Z$)

Time backwards from $T$ to zero

$X_T, \ldots, X_0$. $X_T = P(x, 0)$, $X_0 = L(x, 0)$.

$X_t$: Do $P$ until $t$ then $L$ until zero

$$F(d, t) := \Pr[S_{t-1} = d - 1] - \Pr[S_t = d]$$

$$-F(d, t) = \Pr[S_{t-1} = d + 1] - \Pr[S_t = d]$$

$$X_{t-1} - X_t = \sum_d A(x - d, t)$$

$A(x - d, t)$ \begin{align*}
&\equiv 0 & \text{if even chips at } w \\
&\equiv F(d, t) & \text{if odd chips, } \rightarrow \\
&\equiv -F(d, t) & \text{if odd chips, } \leftarrow
\end{align*}
\[ X_0 - X_T = \sum_t \sum_w A(x - d, t) = \sum_w \sum_t A(x - d, t) \]

Fixing \( d \)

\( F(d, t) \) unimodal, same sign

\[ \Rightarrow |\sum_t A(x - d, t)| \leq \max_t |F(d, t)| = O(d^{-2}) \]

\[ X_0 - X_T \leq \sum_d O(d^{-2}) = O(1) \]