1. Extremal graph theory

Extremal graph theory and *Ramsey theory* were among the early and fast developing branches of 20th century graph theory. In the introduction we shall begin with describing and proving some *sharp* theorems (Turán's theorem and Kővári-T. Sós theorem) and then the asymptototic structure of the extremal graphs (the Erdős–Simonovits theory). The introductory part will include description of connections to Ramsey theory, topology, number theory, and also some connections to geometry and hypergraph theory.

This historical introduction will be followed by the classification of ordinary extremal graph problems, and the possibility of reducing most non-degenerate extremal graph problems to degenerate extremal graph problems. This and all the subsequent parts will include some important unsolved problems. Thus, e.g., first part will be closed with giving the simplest versions of Turán's famous hypergraph extremal problem, and some important related results.

The series of lectures covers the following areas:

1. Introduction, history and some central theorems
   (a) Ramsey theory (Erdős-Szekeres)
   (b) Extremal graph theory (Turán/Erdős)
2. Classification of extremal graph problems and lower bound constructions
   (a) The asymptotic structure of extremal graphs
   (b) Degenerate extremal graph problems
   (c) Lower bound constructions using random graphs and finite geometries
3. Regularity lemma for graphs
   (a) Origins/connections to the existence of arithmetic progressions in dense sequences
   (b) Connection to the quantitative Erdős–Stone theorem
   (c) First graph theoretic applications (Ruzsa–Szemerédi theorem, Ramsey-Turán problems)
   (d) Counting lemma, removal lemma, coloured regularity lemma
4. Some recent results
   (a) 3-colored Ramsey theoerem for cycles
   (b) Erdős-Sós conjecture on trees

2. Classical extremal graph theory

Extremal graph theory has several roots:applications in geometry, logic (Ramsey theory) and number theory, ...

The first result in extremal graph theory is Mantel's theorem, which is Turán's theorem for K_3, see below. Next Erdős and Szekeres (re)invented Ramsey's theorem (and later they realized that it was proved a little earlier by Ramsey). Turán wanted to replace the condition of Ramsey's theorem by conditions on the edge-number: Let $T_{n,p}$ be the p-class Turán graph: n vertices are partitioned into p classes as uniformly as possible and two vertices are joined iff they belong to different classes.

Theorem 1 (Turán, 1940)**.** *Among all the graphs G_n not containing K_{p+1} there is only one having maximum number of edges, namely, the Turán graph $T_{n,p}$.*

Erdős should have "invented extremal graph theory in 1938," but he missed it. It was important, that Turán, proving his theorem also asked several related questions.

- What is the situation for other excluded subgraphs. e.g., for the graphs of the 5 Platonic bodies?
- What is the situation for the path, loops?

This leads, among others, to the Erdős-Gallai theorem: Given a family $\mathcal{L}$ fo (forbidden) graphs, denote by $\mathbf{ex}(n, \mathcal{L})$ the maximum number of edges a graph G_n on n vertices can have without containing subgraphs from $\mathcal{L}$.

Theorem 2 (Erdős-Gallai)**.** *If P_k is a path of k vertices,* $\mathbf{ex}(n, P_k) = \frac{k-2}{2}n + O(1)$.

Later Turán conjectured that $\sqrt{n}$ is the Ramsey number: all the graphs on n vertices contain either a complete graph of $\lfloor\sqrt{n}\rfloor$ vertices or an empty graph of this many vertices. Erdős pointed out that the right order of magnitude is not $\sqrt{n}$ but $c \log n$. This is how the random graph method emerged.

Questions from topology led to the Erdős–Stone theorem (1946) and to its important very special case, with much better estimates:

Theorem 3 (Kővári-T. Sós-Turán)**.**

$$\mathbf{ex}(n, K(a, b)) \le c_{a,b} n^{2-(1/a)}.$$

Some lower bounds were obtained (a) using graphs defined via finite geometry, (b) using random graphs, (c) using high dimensional geometric constructions, and (d) using more advanced methods from commutative algebra.

3. Regularity lemma

The famous Erdős-Turán problem on arithmetic progressions in dense sequences of integers led to a complicated version of the *Szemerédi regularity lemma.* Later

(1976) some investigations on the quantitative Erdős–Stone theorem resulted in the modern form of the regularity lemma, one of the most powerful methods in asymptotic graph theory. This had many applications, among others, for quasi-random graphs.

The regularity lemma asserts that all graphs can be approximated – in some sense – by a generalized random graph:

Definition 4 (Generalized random graph)**.** Given an $r \times r$ matrix $A = (p_{ij})$ where p_{ij} are probabilities, for $x_1, \dots, x_r$ we define the corresponding generalized random graph by fixing r groups of vertices, $C_1, \dots, C_r$ and joining each vertex of C_i to each (other) vertex of C_j with probability p_{ij}, independently.

Definition 5 (Regularity condition)**.** Given a graph G_n and two disjoint vertex sets $X \subseteq V$, $Y \subseteq V$, we shall call the pair (X, Y) ε–regular, if for every $X^* \subset X$ and $Y^* \subset Y$ satisfying $|X^*| > \varepsilon|X|$ and $|Y^*| > \varepsilon|Y|$,

$$|d(X^*, Y^*) - d(X, Y)| < \varepsilon.$$

Theorem 6 (Regularity Lemma 1976)**.** *For every $\varepsilon > 0$ and ν_0 there exists a $\nu(\varepsilon, \nu_0)$ such that for every G_n, $V(G_n)$ can be partitioned into ν sets $U_0, U_1, \dots, U_\nu$, for some $\nu_0 < \nu < \nu(\varepsilon, \nu_0)$, so that $|U_i| = m$ or $U_i = m+1 \approx n/\nu$ for every $i > 0$, and for all but at most $\varepsilon\binom{\nu}{2}$ pairs (i, j) (U_i, U_j) is ε–regular. (U_0 is discarded.)*

4. Some recent results

In the last lectures we shall sketch proofs of some recent results, among others, the sharp version of Łuczak's theorem on the Bondy–Erdős conjecture:

Theorem 7 (Kohayakawa–Simonovits–Skokan)**.** *If C_n is an odd cycle of n vertices and $R(C_n, C_n, C_n)$ denotes the three-color Ramsey function, then for $n > n_0$ $R(C_n, C_n, C_n) = 4n - 3$.*

We shall also sketch the proofs of several further results and recent progress on the Erdős–Sós conjecture on trees.