Exercise 1. *Find nice classes of balanced graphs.*

Exercise 2. *Show that* $p_0 = n^{-1/m(H)}$ *is the threshold function for* $H \subset G(n,p)$*, where* $m(H) = \max\{e(J)/v(J) \colon J \subset H, |V(J)| > 0\}$.

Exercise 3. *Show that if* $p = Cn^{-2/3}$ *and* C *is a large enough constant, then almost every* $G(n,p)$ *is such that* $G(n,p) \to (K^3)^v_2$*, that is, any colouring of the vertices of* $G(n,p)$ *with* 2 *colours necessarily contains a monochromatic* K^3.

Generalize it from K^3 *to arbitrary graphs* H *and to more than* 2 *colours.*

Exercise 4. *How do the probabilities* $\mathbb{P}(G(n,p)$ *is Hamiltonian*$)$ *and* $\mathbb{P}(G(n,p)$ *is Hamiltonian* $\mid G(n,p)$ *is planar*$)$ *compare?*

Exercise 5. *Let* H *be a graph and* X *be the number of labelled copies of* H *in* $G(n,p)$*. Let* $X_{H'}$ *be the indicator r.v. that* H' *spans a copy of* H*, where* H' *is a labelled subset of cardinality* $v(H)$ *in* $[n]$*. Let* $\Delta^* = \sum_{(H',H'')} \mathbb{E}(X_{H'}X_{H''})$*, where the sum is over all pairs* (H',H'') *of copies of* H *with at least one common edge. Set* $\Phi_H = \Phi_H(n,p) = \min\{\mathbb{E}(\#\{J \hookrightarrow G(n,p)\}) \colon J \subset H, |E(J)| > 0\}$ *and* $\mu = \mathbb{E}(X) = \mathbb{E}(\#\{H \hookrightarrow G(n,p)\})$*. Verify*

(*i*) $\Delta^* = \Theta(\mu^2/\Phi_H)$ *and*

(*ii*) $\mathrm{Var}(X) = \Theta(\mu^2/\Phi_H)$ *if* p *is bounded away from* 1 *(and* $= O(\mu^2/\Phi_H)$ *always).*

Exercise 6. *Deduce from the binomial model that if* $M \geq e(H)$*, then* $\mathbb{P}(H \not\subset G(n,M)) \leq \exp\{-\Theta(\Phi_H)\}$.

Exercise 7. *Suppose the following conjecture is true.*

Conjecture. *Suppose* H *is a bipartite graph. For any* $\beta > 0$*, there is* C_0 *such that for any* $M = M(n)$ *such that* $\Phi_H \geq C_0 M$*, we have*

$$\mathbb{P}(H \not\subset G(n,M)) \leq \beta^M$$

for all large enough n.

(*i*) *Deduce a fault-tolerance result for* $G(n,M)$ *with respect to* H*. Estimate* $f(n,\eta,H) = \min|E(\Gamma)|$*, where* Γ *ranges over all graphs with the property* $\Gamma \to_\eta H$.

(*ii*) *Translate the hypothesis* $\Phi_H \geq C_0 M$ *to something "nicer." Suppose* $|V(H)| > 2$*. Then let* $d_2(H) = \frac{|V(E)|-1}{|V(H)|-2}$*. For* $H = K^1$ *and* $2K^1$ *let* $d_2(H) = 0$*; set* $d_2(K^2) = 1/2$*. Finally, let* $m_2(H) = \max\{d_2(J) \colon J \subset H\}$*. Consider* $M_0 = n^{2-1/m_2(H)}$.

Exercise 8. *Let* $p = \frac{1}{n}(\log n + \log\log n + c_n)$ *and deduce*

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \text{ is Hamiltonian}) = \begin{cases} 0 & \text{if } \lim_n c_n = -\infty, \\ e^{-e^{-c}} & \text{if } \lim_n c_n = c \in \mathbb{R}, \\ 1 & \text{if } \lim_n c_n = \infty. \end{cases}$$

from the theorem that for a.e. $\mathbf{G} = (G_t)_{t=0}^N$*, we have* $\tau(\mathbf{G}, \textsc{Ham}) = \tau(\mathbf{G}, \delta \geq 2)$*, where* τ *denotes the hitting time in a random graph process.*

Exercise 9. *Show that for any integer* $k \geq 1$ *and any real* $\epsilon > 0$*, there is* $C = C(k,\epsilon)$ *such that if* $p = C/n$*, then* $|V(\mathrm{core}_k(G(n,p)))| \geq (1-\epsilon)n$ *almost surely.*

Exercise 10. *Show that almost surely* $G(n,p)$ *has* $\mathrm{circ}(G(n,p)) \geq (1-\epsilon)n$ *if* $pn \geq C_\epsilon$.

Exercise 11. *Prove the following central lemma from Pósa's proof.*

Let $P = x_1x_2\dots x_h$ *be a longest* x_1*-path in a graph* G *and let* U *be the set of right endvertices of the right transforms of* P*. Set*

$$N = \left\{x_i\colon 1 \le i < h, \{x_{i-1}, x_{i+1}\} \cap U \neq \emptyset\right\}$$

and $R = V(P) \setminus (U \cup N)$*. Then the graph* G *contains no* U–R *edge.*

Exercise 12. (*i*) *Deduce the following corollary from Pósa's lemma:*

Let $u \ge 1$ *be an integer. Suppose a graph* G *is such that* $|U \cup \Gamma(U)| \ge 3|U|$ *for all* $U \subset V(G)$ *with* $|U| \le u$*. Then* G *contains a path* P^{3u} *on* $3u$ *vertices.*

(*i*) *Does the same approach yield the bipartite version? I.e., Let* $b \ge 1$ *be an integer. If the bipartite graph* B *is* $(b, 2)$*-expanding, then* B *contains a path* P^{4b} *on* $4b$ *vertices.*

Exercise 13. *Deduce the following results from the result of Alon and Füredi:*

(*i*) *Let* $p > 1/2$*. Then a.e.* $G(2^d, p)$ *contains* Q^d *as a subgraph.*

(*ii*) *Let* L_k *be the* $k \times k$ *square lattice, that is, the graph on the* $(i, j) \in [k] \times [k]$ *with two such pairs joined by an edge if they differ by* 1 *in one coordinate. Find* p_- *such that almost no* $G(k^2, p_-)$ *contains* L_k*. Using Alon–Füredi–theorem, find* p_+ *such that almost every* $G(k^2, p_+)$ *contains* L_k