Exercise 1. *Prove the removal lemma for graphs, i.e., for every graph* F *on* ℓ *vertices and every* $\varepsilon > 0$ *there exist* $\delta > 0$ *and* n_0 *such that every graph* G *on* $n \geq n_0$ *vertices containing only* δn^ℓ *copies of* F *can be made* F*-free by removal of at most* εn^2 *edges.*

- *How should an induced version of this statement look? Can you prove it* ***(very hard)****?*
- *Deduce the Ruzsa–Szemerédi–theorem from the removal lemma for* $F = K_3$.

Exercise 2. *Deduce Roth' (second famous) theorem from the Ruzsa–Szemerédi–theorem.*

Exercise 3. *Prove that if* G *is an* n*-vertex graph which is the union of* n *induced matchings, then* $e(G) = o(n^2)$. *(A matching* $M \subset E(G)$ *is induced in* G *is there is* ***no*** *edge* xy *in* $E(G) \setminus M$ *such that* $xy \subset \bigcup_{e \in M} e$.*)*

Deduce the Ruzsa–Szemerédi-theorem from this matching result.

Exercise 4. *Prove that for every integer* K *and every* $\varepsilon > 0$ *there exist* $c > 0$ *and* n_0 *such that for every graph* G_n *(*$n \geq n_0$*) there exists mutually disjoint sets* $V_1, \ldots, V_k \subset V(G_n)$ *such that for every* $1 \leq i < j \leq k$ *the pair* (V_i, V_j) *is* ε*-reglur and*

- *either* $d(V_i, V_j) < 1/2$ *for* ***all*** $1 \leq i < j \leq k$
- *or* $d(V_i, V_j) \geq 1/2$ *for* ***all*** $1 \leq i < j \leq k$.

Exercise 5. *Show that for any graph* L *which is not a forest and any rooted tree* T *the following holds for sufficiently large* n*:*

$$\mathbf{ex}(n, L) = \mathbf{ex}(n, M)\,,$$

where M *is obtained by attaching a copy of* T *to some vertex of* L.

Exercise 6. *Suppose* G_n *contains no copy of* C_6. *Show that* G_n *contains a subgraph with at least half of the edges which contains no* C_4.

Exercise 7. *Show that* $\mathbf{ex}(n, C_4) = n^{3/2}/2 + o(n^{3/2})$.

Exercise 8. *Let* $\mathcal{L}$ *be a finite family of graphs. Show that* $\mathbf{ex}(n, \mathcal{L}) = o(n^2)$ *iff* $\mathcal{L}$ *contains a bipartite graph. Is the finiteness of* $\mathcal{L}$ *needed?*

Exercise 9. *Show that for all integers* $2 \leq p \leq q$ *there exist* $C > 0$, $c > 0$ *and* n_0 *such that every graph* G_n *on* $n \geq n_0$ *vertices with* $e(G_n) \geq Cn^{2-1/p}$ *contains at least*

$$c\left(\frac{e(G_n)}{n^2}\right)^{pq} n^{p+q}$$

copies of $K_{p,q}$.

Exercise 10. *For integers* $k \geq 2$ *and* ℓ *let* $K^{(k)}(\ell)$ *denote the complete* k*-partite* k*-uniform hypergraph with* ℓ *vertices on each partition class. Show that*

(*i*) *there exist* $C > 0$ *such that for sufficiently large* n *we have* $\mathbf{ex}(n, K^{(k)}(\ell)) \leq Cn^{k-1/(\ell^{k-1})}$,

(*ii*) *For all integers* $k \geq 2$ *and* ℓ *and* $c' > 0$ *there exist* $c > 0$ *such that every sufficiently large* n*-vertex* k*-uniform hypergraph with* $c'n^k$ *edges contains at least* $cn^{k\ell}$ *copies of* $K^{(k)}(\ell)$.